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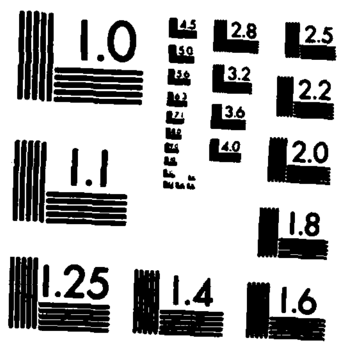

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INTERDEPENDENCE OF EQUIVALENT STEADY STRESS, CRACK GROWTH AND FAILURE ON SEQUENCE AND AMPLITUDE OF IRREGULAR LOADING

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ABSTRACT

This paper shows how to predict the minimum or other life of material limited by cyclic crack growth and crack failure as a function of the different sequences of amplitudes of loading that may occur under random, quasi-random, or controllable loading conditions. It is assumed that the incremental growth, as well as the criterion of failure, is independent of the history of loading and environment. The method covers the most commonly used law of crack growth and of crack failure. It is obviously applicable to other cyclically induced cumulative phenomena, e.g., degradation as in erosion or thermal fatigue.

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INTRODUCTION

If one accepts some of the most common types of laws for rate of crack growth and crack failure, including the assumption that incremental growth is independent of the history of loading, one is in a position to address himself to some important questions of general behavior, including that of the effect on crack growth and life of different types of loads.

In particular, we are concerned in this paper with the effect of load sequence on crack growth and failure, in cases where the loading is a spectrum of loads of different amplitudes. To take an example, let us suppose we have a gun or other tube, containing a small internal crack, to which a series of internal pressures, some high, some low, are to be applied. We may ask, will the ultimate crack growth be less if the high pressures are applied first, when the crack is shortest, than if they are applied last, when the crack is longer? And, since crack length and stress may be simultaneously significant in cataclysmic, e.g., K_{IC} failure, what sequence of loads will give the shortest life? And is this serious condition likely to occur under random loading?

Also, are there circumstances such that for growth under many cycles of loading, sequence of loading is unimportant? If so, what is the equivalent steady stress that gives the same growth per cycle of stressing as does a spectrum of loads? What do we mean by the term cycle of stressing for an irregular spectrum?

It is to such questions that this paper is addressed. Our treatment has been heuristic rather than mathematically complete; it is hoped that a method of treatment of such questions has been made sufficiently self-evident that the reader can use it for himself, for his own particular problems, if he so chooses.

CONDITION FOR MAXIMUM AND MINIMUM CRACK GROWTH

The growth Δa, of a crack of length "a", per cycle of nominal loading stress, S, is commonly given by

$$\Delta a/\text{cycle} = C_1 K^m = C_2 S^m a^{m/2} Y_c{}^m, \qquad m \geq 2$$

in regions of K or "a" where the C's are constants and K is the toughness stress intensity parameter. S as used in the above formula is usually taken to be $S_{max}-S_{min}$, where S_{max} is the maximum and S_{min} the minimum stress in a cycle of stressing. If the loading spectrum is irregular in shape, <u>we define each rise in stress as a cycle</u>. Y_c is a parameter, constant or increasing* with increase in "a" and expressing the geometry involved, such as 1.1215 for an edge crack or $1/\sqrt{\cos \pi a/w}$ for a central crack in a plate of width w. We assume that the constants C_1 and C_2 may be different in different regions of crack length and, correspondingly, that there may be sudden transitions in the rate of crack growth as a crack grows longer. Though we have some exceptional evidence of sudden decreases in rate, this finding should be further investigated, and we shall assume here that the rate does <u>not</u> decrease with crack growth. Furthermore, we restrict ourselves to cases where the loading stresses are tensile, thus avoiding the complication of complete crack closure by compression and ambiguous experimental results.

Thus, for N stresses, S $(= S_{max}-S_{min})$

$$\sum \frac{\Delta a}{a^{m/2} Y_c^m} = C_2 \sum S^m$$

$\sum$ over N stresses, S

*For cracks leading to failure; this restriction eliminates cases of crack closure resulting from crack growth toward a negatively stressed region.

and the sum on the left must increase, as will "a" when the sum on the right increases with additional cycles of S.*

Thus "a" will be greater at any stage of crack growth, the larger the sum on the right. And the latter sum will obviously be greatest if the largest values of S, occurring in the entire spectrum of loading, are crowded into it (independent of order, so long as they are the largest). Therefore, with the above law of crack growth "a" will be maximum up to any number of cycles, N, if the largest occurring values of S are used up, in computing the sum up to the N stresses, on the right. And if, on the other hand, the smallest values of S are used first, "a" will be minimum.

The intermediate cases of crack growth for n cycles may be found, in order of severity, by progressively decreasing the sum of the n stresses on the right. The probability of occurrence of any number n of selected loads out of a total of N occurring in a representative† cyclic graph of loading is

$$P = \frac{n!\ (N-n)!}{N!} \quad .$$

The final value of "a", i.e., the total growth corresponding to use of all the cycles of S, is a fixed value, independent of the order of application of the stresses, S.

*In conformance with restrictions made up to this point, it will be observed that we confine ourselves to the case where the increments are linearly additive. That is, we confine ourselves to laws of growth for which the incremental growth per load application is independent of the history of loading and environment. Later on it may be noted that we have implicitly restricted our failure criteria in the same way.

†"Representative" here means long enough to determine frequencies of occurrence of all loads. Although the loading spectrum of a machine under test may be controlled, in practice one is interested in ensemble characteristics.

Figure 1 illustrates by a simple example the above formula for an initial crack length a = 1 and a loading stress sequence of three cycles, S = 1, 2, 3. (The stresses and lengths may be considered to be measured in any units the reader desires, e.g., S in ksi, "a" in inches.) It demonstrates the method of computation for "a" as well as the $\Delta a/a$ of the above equation, and shows the equality of both sides of the equation.

Our argument above will perhaps be made clearer by considering the above growth law expressed in integral form. This representation is possible because the steps Δa are very small and numerous. We have

$$\int_{a_o}^{a} \frac{da}{a^{m/2}\, Y_c^m} = C_2 \int_0^N S^m \, dN$$

Suppose, for simplicity Y_c is constant.

Then

if m>2

$$\frac{1 - (a_o/a)^{m/2-1}}{Y_c^m \, a_o^{m/2-1} (m/2 - 1)} = C_2 \int_0^N S^m \, dN$$

if m=2

$$Y_c^{-2} \ln a/a_o = C_2 \int_0^N S^2 \, dN$$

Here "a" is the crack length after any specified number, N, of cycles of stressing.* In either case, m > 2 or m = 2, the larger "a" is, the larger the left hand term is, and this will be greatest when the integral on the right is greatest, that is, when it contains the largest values of S. Thus "a" is maximum when the integral on the right contains the largest values of S.

*So long as "a" remains in a region where C_2 is constant.

	Start	Cycle 1	Cycle 2	Cycle 3
Stress, S	0	1	2	3
S^2	0	1	4	9
ΣS^2	0	1	5	14
$\Delta a=S^2 a$	$\underline{0}$	1x1=1	4x2=8	9x10=90
$a=a_o+\Sigma\Delta a$	1	1+1=2	2+8=10	10+90=100
$\Delta a/a$	0	1/1=1	8/2=4	90/10=9
$\Sigma\Delta a/a$	0	1	5	14

Figure 1. Simple Example. Growth Law Δa/cycle of stressing = $S^2 a$. Crack Length $a = a_o + \Sigma\ S^2 a$ and $\Sigma\ \Delta a/a = \Sigma S^2$ for N = 1, 2, and 3 Cycles of Stressing.

Clearly the order of application of these (N) selected largest stress cycles is of no importance since the integral on the right may be conceived as an area made up of the sum of small rectangular areas of base dN and height S^m, the order of addition being inconsequential.

Abrupt changes in C upon reaching certain crack lengths, representing transitions to higher rates of crack growth as the crack grows longer, do not invalidate the above conclusion. For, obviously, we wish the crack to have a maximum length up to the transition and thereafter we may consider the transition point with its associated crack length, as the start of a new problem. Pushing the transition points closer together, we approach the case of a continuously increasing C.

Analytically, if C is an increasing function of "a", it will go under the summation of integral on the left, as Y_c does, but obviously will not invalidate the above conclusions.

In this respect, it may be satisfying to consider the writer's crack growth law with transitions. The crack growth laws between transitions are like those just considered with m=2 in $K^m \sim S^m a^{m/2}$ but the constants are theoretical. The constant in any region is double that in the preceding region and for constant loading stress, the crack length at any transition is twice the length at the preceding transition, i.e., $\bar{a}_n = 2^n \bar{a}_o$. These crack growth transitions $\bar{a}_n$, $\bar{N}_n$, on an "a" vs N plot fall on a hyperbola

$$N - \bar{N}_o = \frac{2 \ln 2}{C_{th}} (1 - \bar{a}_o/a)$$

where $\bar{a}_o$, $\bar{N}_o$ are values of $\bar{a}_n$, $\bar{N}_n$.

"$\bar{a}_o$" is a length characteristic of the material considered. For some materials the usual initial crack length in common crack growth specimens does not greatly differ from $\bar{a}_o$. In this case $N_o = 0$, of course.

$$C_{th} = C\,S^2 = \left[\left(\frac{2}{3}\right)^{1/3} \left(1-\mu^2\right)^{4/3} \left(\frac{1}{E}\right)^{4/3} \left(\frac{1}{Y}\right)^{2/3}\right] S^2$$

where E is Young's modulus, μ is Poisson's ratio and Y is the yield strength under cyclic loading corresponding to stress* S.

*The above formula for C_{th} is for the usual case of very small scale yielding near the crack tip, thus for small S/Y ($\equiv S_y/Y$), or, more generally, for isotropic loading $S_x = S_y$ with essentially isotropic yielding.

A more generally applicable formula for the case where the loading stress S_x along the crack, and the loading stress S_y perpendicular to the crack, rise and fall together is

$$C_{th} = \frac{(2/3)^{1/3} \; [E/(1-\mu^2)]^{-4/3} \; Y^{-2/3} S_y^{\;2}}{\left\{1 - (\sqrt{3}/2)\; (S_y/Y)\; [(1 - S_x/S_y)]\right\}^{2/3}}$$

(This formula is the same as it was in the corresponding footnote of the first (1977) issue of this report except that a term then supposed to render the formula applicable to orthotropic yielding has been removed.)

For simplicity, it is this hyperbola, which is periodically correct, that we consider as our growth law, instead of the actual law with transitions.

Differentiating N,

$$\frac{dN}{da} = \frac{2 \ln 2}{C_{th}} \frac{\bar{a}_o}{a^2} \quad \text{(S = Const. for each increment and cycle of crack growth.)}$$

Hence

$$\frac{(2\ln 2)\ \bar{a}_o\ \Delta a}{a^2} = C \sum_{N \text{ stresses}} S^2$$

where $C = C_{th}/S^2$ = Const. Or, in integral form,

$$\int_{a_o}^{a} \frac{(2\ln 2)\ \bar{a}_o\ da}{a^2} = C \int_0^N S^2\, dN$$

i.e.,

$$(2\ln 2)\ (\bar{a}_o/a_o)\quad (1 - a_o/a) \quad = C \int_0^N S^2\, dN$$

(Note that m = 4, page 4, also gives an "a" vs N hyperbola of this shape, with stresses to the fourth power instead of second power as here, a difference not noticeable in constant stress tests in which growth rate constants are determined empirically.)

The form of these expressions validates the conclusions already drawn. The maximum crack length up to any number of cycles corresponds to application of the cycles of greatest stress, independent of order.

Figure 2 shows the effect of load sequence by extending the simple example of Figure 1 to all possible sequences of the three loading stresses given there. Since there are three distinct loading stresses, there are six cases in all, S = 1, 2, 3; S = 1, 3, 2; S = 2, 1, 3; S = 2, 3, 1; S = 3, 1, 2; S = 3, 2, 1 for the first, second, and third cycles, respectively. As noted, equal crack lengths result from application of a set of loads independent of order of application. Up to any point, use of the largest loads results in greatest "a" and use of the smallest loads results in least "a".

It is not difficult, in some cases, to satisfactorily approximate a spectrum of numerous loads, which may be applied in any sequence, by a few loads such that the effects and frequency of application of different load sequences may be easily handled and visualized without the aid of a large computer. Thus, for example, since d ln a $\sim S^m$ dN with Y_c = Const., in the rate of crack growth formula, if there are n_1, n_2 ... n_o numbers of stresses S_1, S_2 ... S_o, respectively, we might form the products $n_1 S_1^m$, $n_2 S_2^m$ ..... $n_o S_o^m$, then omit any stress levels for which any of these products are unimportant compared to the remainder, then lump the remaining stress levels into a few which are applied for the total number of cycles contained in their components, e.g., for stress level p, of these few

$$S_p^m = \frac{n_x S_x^m + n_{x+1} S^m + \text{etc.}}{n_x + n_{x+1} + \text{etc.}}$$

applied for the number of cycles n_p equal to the sum in the denominator of the right-hand side.

Figure 3a is an illustration of crack growth where stresses are applied repetitively, for an m = 2 crack growth equation.

		Start	Cycle 1	Cycle 2	Cycle 3
1.	Stress, S	0	1	2	3
	$\Delta a = S^2 a$	0	(1)(1)=1	(4)(2)=8	(9)(10)=90
	$a = a_o + \Sigma\Delta a$	1	1+1=2	2+8=10	10+90=100
2.	S	0	1	3	2
	Δa	0	(1)(1)=1	(9)(2)=18	(4)(20)=80
	a	1	2	20	100
3.	S	0	2	1	3
	Δa	0	(4)(1)=4	(1)(5)=5	(9)(10)=90
	a	1	5	10	100
4.	S	0	2	3	1
	Δa	0	(4)(1)=4	(9)(5)=45	(1)(50)=50
	a	1	5	50	100
5.	S	0	3	1	2
	Δa	0	(9)(1)=9	(1)(10)=10	(4)(20)=80
	a	1	10	20	100
6.	S	0	3	2	1
	Δa	0	(9)(1)=9	(4)(10)=40	(1)(50)=50
	a	1	10	50	100

Figure 2. Example: Effect of Sequence of Loading Stress S on Crack Growth Δa/cycle of Stressing=$S^2 a$. Initial (S=0) Crack Length, a=1. S=1, 2, and 3. Note Independence of Order of Application, e.g., a=100 for S=1, 2, and 3 in all cases and a=10 for two cycles, S=1 and 2, Cases 1 and 3.

EQUIVALENT STEADY STRESS

By the term equivalent steady stress applied to a cracked specimen, we mean a cyclic stress of constant amplitude whose application would result in the same crack growth as that due to any specified portion of a loading spectrum of irregular amplitude, if the number of cycles of load application were the same for each type of loading.

As so defined, it may have different values for portions of the spectrum made up of different loads, but it will have a unique value for the whole spectrum since this includes all the loads.

The expressions for equivalency are apparent from the summation or integral expressions we have already obtained. Since for both the constant amplitude steady and the irregular stress the crack growth is to be the same, the left-hand sides of these expressions must be the same whether produced by a steady stress or an irregular stress. Thus if there are N_r cycles in the portion of the spectrum being considered, and if the steady equivalent stress is S_{eq}, the sum or integral on the right must be equal to $S_{eq}^m N_r$.

That is,

$$S_{eq}^m N_r = \int_0^{N_r} S^m \, dN = \sum_{N_r \text{ stresses, } S.} S^m$$

for the power laws; m = 2 for the envelope hyperbola.

Thus if there are n_1 stresses S_1, n_2 stresses S_2, ... n_r stresses S_r, such that

$$\sum_{i=1}^{i=r} n_i = N_r \; ; \quad n_i/N_r = f_i$$

Part A

With $\frac{da}{dN} = S^2 a/100$

$\ln a_1/a_o = S_1^2 n_1/100$ $\ln a_2/a_1 = S_2^2 n_2/100$ $\ln a_3/a_2 = S_3^2 n_3/100$

$\ln a_2/a_o = \ln a_1/a_o + \ln a_2/a_1$ $\ln a_3/a_o = \ln a_2/a_o + \ln a_3/a_2$

Region, i,	0 (Start)	1	2	3
Cycles, n_i	0	10	2	1
S_i	0	1	2	3
$S_i^2 n_i/100$	0	0.10	0.08	0.09
$\Sigma S_1^2 n_i/100$	0	0.10	0.18	0.27
$\ln a_i/a_{i-1} (=S_i^2 n_i/100)$	0	0.10	0.08	0.09
$\Sigma \ln a_i/a_{i-1} (=\Sigma S_i^2 n_i/100)$	0	$\ln a_1/a_o = 0.10$	$\ln a_2/a_o = 0.18$	$\ln a_3/a_o = 0.27$
a_i	$a_o = 1$	$a_1 = 1.11$	$a_2 = 1.20$	$a_3 = 1.31$

Part B

Let $\Sigma n_i = N_t (=13)$; $n_i/N_t = f_i$

$$\ln a_3/a_o = (S_1^2 n_1 + S_2^2 n_2 + S_3^2 n_3)/100$$

$$= (S_1^2 f_1 + S_2^2 f_2 + S_3^2 f_3) N_t/100 = S_{eq}^2 N_t/100$$

Thus $S_{eq}^2 = S_1^2 f_1 + S_2^2 f_2 + S_3^2 f_3 = 1\ (10/13) + 4(2/13) + 9\ (1/13) = 27/13$

$$S_{eq} = 1.44$$

Figure 3. Example with m=2, $da/dN=S^2a/100$. Amplitude (Part A) and Equivalent Steady Stress (Part B) when n_1 Cycles of S_1, n_2 Cycles of S_2, and n_3 Cycles of S_3 are applied in succession.

we have*

$$S_{eq}^{m} N_r = n_1 S_1^m + n_2 S_2^m + . . . + n_r S_r^m$$

$$= (f_1 S_1^m + f_2 S_2^m + . . . + f_r S_r^m) N_r$$

whence

$$S_{eq}^{m} = f_1 S_1^m + f_2 S_2^m + . . . + f_r S_r^m$$

in which m = 2 for the hyberbola envelope law as well as the m = 2 power law.

Figure 3b shows the calculation for S_{eq} for the simple case of Figure 3a, where three loads were applied in succession for different numbers of cycles.

For another example, suppose that the distribution of frequencies is Gaussian in N_r cycles, σ being the standard deviation.

$$S_{eq}^{m} N_r = \int S^m \frac{N_r}{\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{S-S_\mu}{\sigma}\right)^2\right] d\left(\frac{S-S_\mu}{\sigma}\right)$$

where S_μ is the mean value of the cyclic stress rise $S = S_{max} - S_{min}$, so that

$S = \left[\left(S - S_\mu\right) + S_\mu\right]$ and thus

$$S_{eq}^{m} = \frac{\sigma^m}{\sqrt{2\pi}} \int \left[\frac{S-S_\mu}{\sigma} + \frac{S_\mu}{\sigma}\right]^m \exp\left[-\frac{1}{2}\left(\frac{S-S_\mu}{\sigma}\right)^2\right] d\left(\frac{S-S_\mu}{\sigma}\right)$$

$$= \frac{\sigma^m}{\sqrt{2\pi}} \int \left(X + \frac{S_\mu}{\sigma}\right)^m \exp\left(-\frac{X^2}{2}\right) dX$$

*See Appendix for a cyclic relationship in lieu of equivalent stress.

where $X \equiv (S-S\mu)/\sigma$

Since the lower limit (least value) of S is zero, the lower limit of X is $-(S_\mu/\sigma)$.

If this and the upper limit are large, we may assume the integration is between $-\infty$ and $+\infty$.

Let us assume this is the case and that <u>m = 2</u>. Then

$$S_{eq}^2 = \frac{\sigma^2}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \left[X^2 + 2X\frac{S_\mu}{\sigma} + \left(\frac{S_\mu}{\sigma}\right)^2 \right] \exp\left(-\frac{X^2}{2}\right) dX$$

The exponential function is positive and symmetrical about $X = 0$ so that the middle term in the brackets makes no contribution and we have

$$S_{eq}^2 = \sigma^2 + S_\mu^2 \qquad \xleftarrow{\text{"Gaussian } S_{eq}\text{"}}$$

Naturally, if σ is zero, there is no scatter and $S_{eq} = S_\mu$

The case of the parabolic distribution, with its finite cut-off, is treated for general values of m as part of the problem of probability of failure.

PROBABILITY OF FAILURE

In this section we consider, in an illustrative way, the probability of failure of a piece of material containing a crack or notch and subjected to the various succession of loads of a given spectrum of loading. It is presumed that the piece will fail when the loading stress reaches a critical value, S_{mc}. In current cases of interest concerning cracks, this value of S_{mc} is commonly obtained from an association of critical toughness value, K_{IC}, and crack length. The latter depends on the number of cycles

of loading together with the equivalent steady stress to which the piece has been subjected. The practical limitations of this approach to S_{mc} and probability of failure will be briefly considered along with notch failure.

We first consider the case of random loading and then a class of ordered loading. In the latter case we consider, in particular, the effect of the condition for maximum crack growth.

1. Random Loading

Let us assume that the distribution of both the maximum stresses S_m, $S_m \equiv S_{max}$, and the stress rises, $S = S_{max} - S_{min}$ are parabolic* with the same standard deviation σ. Let $S_{m\mu}$ be the mean value of S_{max} and S_μ be the mean value of the rises, S.

Thus, for the maximum stresses

$$f = \frac{3\sqrt{5}}{20\sigma}\left[1 - \frac{(S_m - S_{m\mu})^2}{5\sigma^2}\right], \quad -\sigma\sqrt{5} \le (S_m - S_{m\mu}) \le \sigma\sqrt{5}$$

with $f = 0$ for $S_m \ge \sigma\sqrt{5}$ and for $S_m \le -\sigma\sqrt{5}$

so that the probability that $S_m \ge S_{mc}$ is

$$P\,(S_m \ge S_{mc}) = \int_{S_{mc}-S_{m\mu}}^{\sigma\sqrt{5}} \frac{3\sqrt{5}}{20\sigma}\left[1 - \frac{(S_m - S_{m\mu})^2}{5\sigma^2}\right] d\,(S_m - S_{m\mu})$$

$$= (3\sqrt{5}/20)\left\{2\sqrt{5}/3 - \left[(S_{mc} - S_{m\mu})/\sigma\right]\left[1 - (1/15)(S_{mc} - S_{m\mu})^2/\sigma^2\right]\right\}$$

*Note that the constants in the expressions used below are not the same as that of the peak region of the Gaussian distribution. Here σ is the σ of the parabolic distribution, not the σ of the Gaussian.

for $-\sigma\sqrt{5} \leq (S_m - S_{m\mu}) \leq \sigma\sqrt{5}$,

with $P = 0$ for $S_m \geq \sigma\sqrt{5}$ and for $S_m \leq -\sigma\sqrt{5}$.

Similarly for the stress differences, $S = S_{max} - S_{min}$,

$$f = \frac{3\sqrt{5}}{20\sigma}\left[1 - \frac{(S-S_\mu)^2}{5\sigma^2}\right], \quad -\sigma\sqrt{5} \leq (S-S_\mu) \leq \sigma\sqrt{5}$$

with $f = 0$ for $S \geq \sigma\sqrt{5}$ and for $S \leq -\sigma\sqrt{5}$,

so that the equivalent stress, S_{eq}, is given by

$$S_{eq}^m N_r = \int_{-\sigma\sqrt{5}}^{\sigma\sqrt{5}} S^m N_r \left(\frac{3\sqrt{5}}{20\sigma}\right)\left[1 - \frac{(S-S_\mu)^2}{5\sigma^2}\right] d(S-S_\mu)$$

where the crack growth per cycle is proportional to S^m.

Since

$$S^m = (X+S_\mu)^m \quad \text{with } X = S-S_\mu,$$

$$S_{eq}^m = \frac{3\sqrt{5}}{20\sigma}\int_{-\sigma\sqrt{5}}^{+\sigma\sqrt{5}} (X+S_\mu)^m \left[1 - \frac{X^2}{5\sigma^2}\right] dX$$

$$= \frac{\delta(m+2)\left[(\delta+\tau)^{m+2} + (-\delta+\tau)^{m+2}\right] - \tau\left[(\delta+\tau)^{m+2} - (-\delta+\tau)^{m+2}\right]}{\delta^3\,(m+1)(m+2)(m+3)(2/3)}$$

where $\delta = \sigma\sqrt{5}$ and $\tau = S_\mu$.

Thus* if

$$m = 1,\ S_{eq} = S_\mu$$

$$m = 2,\ S_{eq}^2 = S_\mu^2 + \sigma^2$$

$$m = 3,\ S_{eq}^3 = S_\mu^3 + 3S_\mu\sigma^2$$

$$m = 4,\ S_{eq}^4 = S_\mu^4 + 6\ S_\mu^2\sigma^2 + (15/7)\ \sigma^4$$

The form of the expression for S_{eq} for m = 2 is the same as it is for the Gaussian distribution which we derived heretofore even though the approximate form of the peak region of the Gaussian, which is also parabolic, has different constants.

Knowing S_{eq} from this formula we can compute the crack length after any number of cycles of loading prior to failure. If m = 2 we may, for example, have the writer's crack growth law, or his periodically correct hyperbolic simplification (page 7) of it. In the latter case,†

---

* $(\delta+\tau)^{m+2} = \Sigma O + \Sigma E$ by binomial expansion

where $\Sigma O = \Sigma$ 1st, 3rd, 5th, etc., terms

$\Sigma E = \Sigma$ 2nd, 4th, 6th, etc., terms

whence For even m

$$S_{eq}^m = \frac{3[\delta(m+2)\Sigma O - \tau\Sigma E]}{\delta^3(m+1)(m+2)(m+3)}$$

For odd m

$$S_{eq}^m = \frac{3[\delta(m+2)\Sigma E - \tau\Sigma O]}{\delta^3(m+1)(m+2)(m+3)}$$

---

† (1) Let it take N_o cycles to get to a_o

$$N_o - \bar{N}_o = (2\ \ln\ 2/C_{th})(1 - \bar{a}_o/a_o) \quad \text{(v.: p. 6)}$$

(2) Let it take $(N + N_o)$ cycles to get to "a" (changing the meaning of N on page 6)

$$(N + N_o) - \bar{N}_o = (2\ \ln\ 2/C_{th})(1 - \bar{a}_o/a)$$

(3) Subtract (1) from (2), above, to get the N cycles it takes to get from a_o to "a."

$$N = -(2\ \ln\ 2/C_{th})(\bar{a}_o/a - \bar{a}_o/a_o)$$

which gives

$$a = a_o/\left[1 - (a_o/\bar{a}_o)(C_{th}N)/(2\ \ln 2)\right] \quad \text{(v.: p. 7)}$$

in all of which $C_{th} = C\ S_{eq}^2$

$$a = a_o / [1 - (a_o / \bar{a}_o)(C\, S_{eq}^{\,2}\, N)/(2 \ln 2)]$$

where $\bar{a}_o$ is a material constant ($\sim\sim 2$ mm), a_o is the measure of the initial crack length (half length of a central crack in a wide plate) and S_{eq} is the equivalent range of loading stress acting perpendicularly to the crack, derived above for the parabolic distribution.

If a critical stress intensity factor, K_{IC}, is applicable to the material used and is known for the temperature where failure is to be evaluated,

$$K_{IC} = (S_{mc} \sqrt{a})\, Y_o$$

$$\text{i.e.,} \quad S_{mc} = K_{IC}/(\sqrt{a}\, Y_c)$$

where S_{mc} is the peak loading stress corresponding to K_{IC}, Y_c is a parameter which is constant or increases for cracks leading to failure, and "a" is a crack length or depth which we find from our crack growth formula. Thus

$$S_{mc} = (K_{IC}/Y_c) / \left\{ a_o / [1 - (a_o/\bar{a}_o)(C\, S_{eq}^2 N)/2 \ln 2)] \right\}$$

and our probability of failure is given by our formula for P, that is, the probability of occurrence of a stress equal to or greater than S_{mc}:

$$P(S_m \geq S_{mc}) = (3\sqrt{5}/20) \left\{ 2\sqrt{5}/3 - [(S_{mc} - S_{m\mu})/\sigma][1 - (1/15)(S_{mc} - S_{m\mu})^2/\sigma^2] \right\}$$

where $S_{m\mu}$ is the mean value the peak stresses.

For $(S_{mc} - S_{m\mu}) \geq \sigma \sqrt{5}$, the probability P is zero. Since S_{mc} is the value of $K_{IC}/(\sqrt{a}\, Y_c)$ it is not a loading stress per se, but is the stress required to cause fracture, a stress value which the loading stress may or may not equal. Thus S_{mc} may exceed any loading stresses applied in our bounded loading spectrum in which case there is zero probability of the occurrence of any loading stress as large as S_{mc}.

As the crack length measure a, increases with increase in N, or K_{IC} decreases with (ordinarily) decrease in temperature, $S_{mc} = K_{IC}/\sqrt{a}$ may decrease from a safe high value until it lies within the loading range, in which case there will be a probability of failure given by the above formula for P. If S_{mc} falls so that $S_{mc}-S_{m\mu} = -\sigma\sqrt{5}$, P = 1 and failure is certain.

In fact K_{IC} may fall so far and so rapidly with temperature decrease on the lower side of the toughness transition temperature that S_{mc} is practically independent of changes in length a. In this case S_{mc} may be lower than the operating range of stresses and the failure may be regarded as a stress failure independent of crack length, associated with operation below the transition temperature, rather than as a K_{IC} failure. Failures insensitive to crack length, but responsive to stress and transition temperature have been observed and studied by this writer.

On the other hand, there are cases where the crack growth necessary to bring S_{mc} down to the operating range of stress is so great that only a practically insignificant amount of life would remain even if K_{IC} failure did not intervene. In practice, through-the-section yielding may even occur in the net section below the crack, before a critical failure stress is attained, such that the resultant deformation may render the piece useless. In this case, however, the critical stress S_{mc} would not be found from the usual formula for K_{IC} which presuppose small scale yielding.

Fatigue life may, of course, be estimated by use of the equivalent steady stress if one employs a fatigue life law conformable in type to that used in our analysis of crack growth. Power law approximations to

any exact law are suitable. Thus if we have a stressed notch to deal with, we may calculate a life to first macroscopic crack appearance and a subsequent life using for crack depth a depth which includes the original notch depth. If appreciable crack growth occurs, the type of analysis for probability of failure with simple cracks which we have illustrated above, would be applicable. The probability of failure prior to cracking may be gotten from the above formula for P, omitting the crack growth equation. S_{mc} in this case might be determined from destructive tests on notched material made after a few cycles of loading, or estimated from the fracture stress F of the material, if this is known, by using theory which takes account of yielding about the notch root.[1,2,3]

2. Ordered Loading

Here we deal with a finite number of loads of different amplitudes, as with a gun that may be loaded with a variety of charges and whose life is limited by crack growth if not terminated by the application of a sufficient load or corresponding K_{IC}. We wish to know what material toughness would be required to withstand any selected load of the loading spectrum, after the application in any sequence, of any other selected loads.

For the most severe condition, these loads would be chosen to give maximum crack growth and the loading stress applied after their application

(1) BEEUWKES, R., Jr. "Characteristics of Crack Failure," Surfaces and Interfaces, Syracuse University Press, Syracuse, NY, Vol. II, 1968, p. 277.
(2) BEEUWKES, R., Jr. "Determination of Fracture Stress and Effective Crack Tip Radius from Toughness (K_{IC}) and Yield Strength (Y)," Army Materials and Mechanics Research Center, AMMRC TR 78-44, October 1978.
(3) TRACEY, DENNIS M., and FREESE, COLIN E. "Cyclic Plasticity Near a Crack-like Elliptical Flaw," Mechanics of Materials, Vol.I, 1982, p. 151-159.

would be chosen to be the maximum loading stress in the entire spectrum of loads. Although this final load may be included among those causing the *maximum crack growth, this inclusion is unimportant where there are many cycles of loading since no appreciable growth occurs in any one loading.

The crack depth resulting from application of any group of loads may be found as discussed in preceding sections. If the final loading stress in the group is S_m, failure by the K_{IC} criterion will occur if

$$(S_m \sqrt{a})\, Y_c \geq K_{IC},$$

assuming K_{IC} is known. Conversely the toughness necessary to prevent failure must be greater than the product on the left-hand side of this relationship.

APPENDIX

Cyclic Relationship in Lieu of Equivalent Stress

We had (top of page 12)

$$S_{eq}^{m} \; N_r = n_1 \, S_1^m + n_2 S_2^m + \, . \; . \; . \, + n_r \; S_r^m$$

with $\Sigma \; n_i = N_r$

Let n_{f1}, n_{f2} etc. be defined by

$$S_1^m \;\; n_{f1} \;=\; S_{eq}^m \;\; N_r$$

$$S_2^m \;\; n_{f2} \;=\; S_{eq}^m \;\; N_r \qquad \text{etc.}$$

Then

$$S_{eq}^m \, N_r = n_1 \; \frac{S_{eq}^m \; N_r}{n_{f1}} \; + \; n_2 \; \frac{S_{eq}^m \; N_r}{n_{f2}} \; + \ldots + \; n_r \; \frac{S_{eq}^m \; N_r}{m_{fr}}$$

whence

$$\sum_{i=1}^{r} \frac{n_i}{n_{fi}} \; = 1$$

of which a particular case may be fatigue provided the n_{fi} correspond to fatigue failures and S_{eq} and N_r to the endurance limit.

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Reinier Beeuwkes, Jr.

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